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COMBINED PERIODIC BOUNDARY-VALUE PROBLEMS AND THEIR APPLICATIONS IN THE THEORY OF ELASTICITY†

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The method described in [1–4] for solving the Hilbert–Riemann boundary value problems posed for a finite number of contours in a complex plane is extended to periodic problems. Mixed periodic problems of the static and stationary dynamic theory of elasticity for an isotropic and orthotropic half-plane and a composite plane that can be reduced to Hilbert–Riemann problems and solved in quadratures, using the method in question, are considered. In particular, problems concerned with a periodic system of punches with sections of complete adhesion and detachment pressing on an elastic half-plane are solved.

1. LET L and M be two systems of smooth contours in the plane of the complex variable $z=x+iy$ that lie in the strip $0 \leqslant \operatorname{Re} z < 2\pi$.

The problem of determining a piecewise analytic periodic function $\Phi(z)$ [5] from the boundary conditions

$$\operatorname{Im}[p^{\pm}(x)\Phi^{\pm}(x)] = f^{\pm}(x), \quad p^{\pm}(x) \neq 0, \quad x \in L \tag{1.1}$$

$$\Phi^{+}(x) = G(x)\Phi^{-}(x) + g(x), \quad x \in M \tag{1.2}$$

extended by periodicity to the whole complex plane with period 2π will be referred to as the combined periodic Hilbert–Riemann boundary-value problem.

Below we shall consider a special case of this problem, which is important in applications. Namely, we shall consider the case when $L = L^1 \cup L^2$, $M = M^1 \cup M^2$, the contour L consists of intervals $\langle a_k, b_k \rangle$ $(k=1, 2, \ldots, N)$, M^1 consists of intervals $[s_k, t_k]$ $(k=1, 2, \ldots, Q)$ on the real axis $0<a_1<b_1<a_2<\ldots<b_N<2\pi$, $0<s_1<t_1<s_2<\ldots<t_Q<2\pi$, $L \cup M = [0, 2\pi)$, $L \cap M = 0$, $G(x) = -G = \text{const}$, with $G>0$ for $x \in M^1$, $G(x)=1$ for $x \in M^2$, and $p^{\pm}(x) = p(x)$ is a real-valued function on L^1 and a purely imaginary function on L^2. Without loss of generality, we can set $p(x) \equiv 1$ for $x \equiv L^1$ and $p(x) \equiv i$ for $x \in L^2$. Following [4], we shall assume that every boundary point of L must belong to L unless it is a boundary point of M^1.

Let each interval $\langle a_k, b_k \rangle$ contain S_k internal nodal points $x = d_{kl}$, where $d_{kl} < d_{k,l+1}$, dividing $\langle a_k, b_k \rangle$ into intervals that belong to L^1 or L^2. The function $p(x)$ has a discontinuity at each of these points. The total number of internal nodes on L is equal to S. The functions $f^{\pm}(x)$ and $g(x)$ satisfy the Hölder condition.

We shall seek a solution of problem (1.1) and (1.2) in the largest class of functions bounded at infinity and integrable at each node of L and M.

Taking into account the automorphy properties, we shall seek the canonical solution of the corresponding homogeneous periodic problem

$$\operatorname{Im}[p(x)\Phi^{\pm}(x)] = 0, \quad x \in L \tag{1.3}$$

$$\Phi^{+}(x) = -G\Phi^{-}(x), \quad x \in M^1 \tag{1.4}$$

in the form

$$X(z) = Z(z)e^{i\psi(z)} \prod_{j=1}^{N} [S(z-b_j)]^{-\alpha_j} \prod_{j=1}^{2K} [S(z-c_j)]^{-\beta_j}, \quad S(z) = \sin\frac{z}{2} \tag{1.5}$$

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Here $Z(z)$ is the canonical solution of the periodic Reimann problem (1.4), c_j are complex numbers such that $0 \leqslant \mathrm{Re}\, c_j < 2\pi$ and $\mathrm{Im}\, c_j \neq 0$; α_j and β_j are integers, $K = E\{\frac{1}{2}N\}$, $E\{t\}$ is the integer part of t, $\psi(z)$ is a solution of the periodic Dirichlet problem

$$\mathrm{Re}\, \psi^{\pm}(x) = h^{\pm}(x), \quad x \in L \tag{1.6}$$

$$h^{\pm}(x) = \pi n_k^{\pm} - \arg Z^{\pm}(x) + \sum_{j=1}^{N} [\alpha_j \arg S(x - b_j) + \beta_j \arg S(x - c_j)]^{\pm} +$$
$$+ \pi m^{\pm}(x) + {}^1/_2\pi(1 - r), \quad x \in L^r \cap \langle a_k, b_k \rangle, \quad k = 1, 2, \ldots, N, \quad r = 1, 2 \tag{1.7}$$

bounded at each of the nodes and at infinity, $n_k^{\pm}$ are integers, $m^{\pm}(x)$ are integer-valued functions which may have jumps at $x = d_{kl} \pm i0$, $m^{\pm}(a_k) = 0$, and $0 \leqslant \arg S(z) \leqslant 2\pi$.

Since $S(z + 2\pi) = -S(z)$, the periodicity condition for $\psi(z)$ imposes certain restrictions on the choice of $Z(z)$ and α_j and β_j. As will be seen later, to construct the general solution of problem (1.1), (1.2) in the required class, one can use the canonical solution with arbitrary behaviour at the nodes. It should be mentioned that, as in [3, 4] and unlike the Riemann problem [5], no canonical solution with given arbitrary behaviour at the nodes can be constructed for the Hilbert–Riemann problem.

Employing the theory of boundary -value problems for automorphic functions [5, 6], the function $Z(z)$, which is periodic along with the argument of its values, can be written in the form

$$Z(z) = \exp\left\{\frac{1}{4\pi i} \int_{M^1} \ln(-G)\, T(t - z)\, dt\right\} = \prod_{j=1}^{Q} \left[\frac{S(z - t_j)}{S(z - s_j)}\right]^{1/2 - i\gamma} \tag{1.8}$$

$$T(z) = \mathrm{ctg}\,({}^1/_2 z), \quad \gamma = {}^1/_2\pi^{-1} \ln G$$

The Dirichlet problem (1.6), (1.7) can now be made periodic if one introduces any closure condition. For example,

$$\beta_{2K} = -\sum_{j=1}^{2K-1} \beta_j - \sum_{j=1}^{N} \alpha_j \tag{1.9}$$

The solution of problem (1.6), (1.7) can also be constructed on the basis of the theory of [5, 6]:

$$\psi(z) = \frac{Y(z)}{4\pi i} \int_L \frac{h^+(t) + h^-(t)}{Y^+(t)} H(t - z)\, dt + \frac{1}{4\pi i} \int [h^+(t) - h^-(t)]\, T(t - z)\, dt \tag{1.10}$$

$$Y(z) = \prod_{k=1}^{N} [S(z - a_k) S(z - b_k)]^{1/2}, \quad Y(z) \sim \exp(\pm {}^1/_2 iNz), \quad z \to \mp i\infty \tag{1.11}$$

$$H(z) = [S(z)]^{-1}, \; N = 2p + 1; \; H(z) = T(z), \; N = 2p;$$
$$p = 0, 1, \ldots \tag{1.12}$$

According to (1.10) and (1.11), if $N > 1$, then the boundedness condition for $\psi(z)$ at infinity leads to the following system of $N - 1$ real equations:

$$\int_L \frac{h^+(t) + h^-(t)}{Y^+(t)} P_n(t)\, dt = 0, \quad n = 1, 2, \ldots, K \tag{1.13}$$

where $P_n(t) = \exp[i(n - \frac{1}{2})t]$ (if N is odd) and $P_n(t) = \exp[i(n-1)t]$ (if N is even), in which the integral numbers $w_j^+ = n_j^+ + n_j^-$ ($j = 1, 2, \ldots, N$) and the complex numbers c_j ($j = 1, 2, \ldots, N - 1$) are unknown. This system has at least one solution [3, 7] such that the affixes of c_j lie on curves that have end-points at a_j, b_j and are contained in one of the half-planes $y \geqslant 0$ or $y \leqslant 0$.

The integers α_j, β_j and $w_j^- = n_j^+ - n_j^-$ as well as the complex number c_N (for an even number N) will remain undetermined for the time being, which gives some freedom in constructing $X(z)$.

We fix the numbers, assuming $n_j^+ = n_j^- = n_j$, $c_N = 0$, $\alpha_j \equiv 0$, and $\beta_j = (-1)^j$, $j = 1, 2, \ldots, 2K$. Then $w_j^+ = 2n_j$ and condition (1.9) is satisfied.

We substitute these numbers into (1.5), (1.7) and (1.10), and we change the variable of integration to pass from integration over the intervals $[a_k, b_k]$ to integration over the arcs of a unit circle in the auxiliary complex plane of $w = e^{iz}$. Following [2], we change the order of integration in the first integral in (1.10). After some elementary transformations, the desired canonical solution takes the form

$$X(z) = e^{i\varphi(z)}e^{i\varphi_0(z)} \prod_{j=1}^{Q} \left[\frac{S(z-t_j)}{S(z-s_j)}\right]^{1/2} \prod_{j=1}^{K} \frac{S(z-c_{2j-1})}{S(z-c_{2j})} \tag{1.14}$$

$$\varphi(z) = -\frac{\gamma Y(z)}{2} \int_{M^1} \frac{H(t-z)\,dt}{Y(t)}$$

$$\varphi_0(z) = \frac{Y(z)}{4\pi i} \int_L \frac{h_0^+(t) + h_0^-(t)}{Y^+(t)} H(t-z)\,dt + \frac{1}{4\pi i} \int_L [h_0^+(t) - h_0^-(t)]\, T(t-z)\,dt$$

$$h_0^\pm(t) = \pi n_k + \sum_{j=1}^{2K} (-1)^j \arg S(t-c_j) + \pi m^\pm(t) + {}^1\!/_2\pi(1-r)$$

$$t \in L^r \cap \langle a_k, b_k \rangle, \quad k = 1, 2, \ldots, N, \quad r = 1, 2$$

Moreover, the system of equations (1.13) also becomes somewhat simpler:

$$\frac{1}{2\pi i} \int_L \frac{h_0^+(t) + h_0^-(t)}{Y^+(t)} P_n(t)\,dt - \gamma \int_{M^1} \frac{P_n(t)\,dt}{Y(t)} = 0, \quad n = 1, 2, \ldots, K \tag{1.15}$$

The function $X(z)$ is periodic, bounded at infinity and has simple poles at $z = c_{2j}$ and zeros at $z = c_{2j-1}$, where $j = 1, 2, \ldots, K$.

The asymptotic form of $X(z)$ at any node can be written as $X(z) = O[(z-d)^\xi]$, where $\zeta = \varepsilon_k$ for $z = s_k$, $\zeta = \delta_k$ for $z = t_k$, $\zeta = \lambda_k$ for $z = a_k$, $\zeta = \nu_k$ for $z = b_k$ and $\zeta = \gamma_{k_l}^{\pm}$ for $z = d_{k_l} \pm i0$.

In the absence of any common points of L and M^1, using estimates of behaviour of the Cauchy-type integrals at the nodes of the lines of integration [5], we get

$$\begin{gathered} \varepsilon_k = -{}^1\!/_2 + i\gamma, \ \delta_k = -{}^1\!/_2 - i\gamma, \ \lambda_k = 0, \ \nu_k = {}^1\!/_2 [m^+(b_k) - \\ - m^-(b_k)] \\ \gamma_{kl}^\pm = \pm [\theta(d_{kl}) + m^\pm(d_{kl} - 0) - m^\pm(d_{kl} + 0)], \ \theta(t) = \\ = \pi^{-1} \arg \{p(t+0)[p(t-0)]^{-1}\} \end{gathered} \tag{1.16}$$

Setting $\gamma_{k_l}^{\pm} = -1/2$ and taking into account that $\theta(d_{k_l}) = \pm 1/2$, $m^\pm(a_k) = 0$ and $m^\pm(d_{k_l}+0) = m^\pm(d_{k,l+1}-0)$, we obtain the following recursion relation for calculating the values of $m^\pm(t)$ on $\langle a_k, b_k \rangle$:

$$m^\pm(d_{kl} + 0) = m^\pm(d_{kl} - 0) + E\{\theta(d_{kl})\} + {}^1\!/_2 \pm {}^1\!/_2$$

From this and (1.16) it follows that $\nu_k = 1/2 S_k$.

If the end-points of any pair of intervals $[s_l, t_l] \subset M^1$ and $\langle a_r, b_r \rangle \subset L$ coincide with one another, then, as in [2], the canonical solution ceases to oscillate in the neighbourhood of the common point of the intervals, and so $\varepsilon_l = \nu_r = 1/2(S_r - 1)$ for $s_l = b_r$, and $\delta_l = \lambda_r = -1/2$ for $t_l = a_r$. The fact that the canonical solution does not oscillate any longer enables one to describe a model of detachment by introducing a section of attachment with slippage adjoining the section of complete adhesion.

Passing to the construction of the general solution of the boundary-value problem in the class of piecewise analytic periodic functions specified above, we consider the function $F(z) = [X(z)]^{-1}\Phi(z)$. This periodic function satisfies the boundary conditions

$$\operatorname{Im} F^{\pm}(x) = f^{\pm}(x)[p(x)X^{\pm}(x)]^{-1}, \ x \in L \tag{1.17}$$

$$F^{+}(x) - F^{-}(x) = g(x)\,[X^{+}(x)]^{-1}, \ x \in M \tag{1.18}$$

and is bounded at infinity.

We set

$$F(z) = F_1(z) + F_2(z), \quad F_1(z) = \frac{1}{4\pi i}\int\limits_M \frac{g(t)}{X^{+}(t)}\, T(t-z)\, dt \tag{1.19}$$

Then $F_2(z)$ is a solution of the periodic Dirichlet problem

$$\operatorname{Im} F_2^{\pm}(x) = f_2^{\pm}(x), \ f_2^{\pm}(x) = f^{\pm}(x)[p(x)X^{\pm}(x)]^{-1} - \operatorname{Im} F_1(x), \ x \in L \tag{1.20}$$

It can also be constructed on the basis of the theory of boundary-value problems for automorphic functions. By analogy with [3], taking into account that the function has poles with unknown principal parts at $z = c_{2k-1}$ $(k = 1, 2, \ldots, K)$ and $z = t_k$ $(k = 1, 2, \ldots, Q)$, we find that

$$F_2(z) = \frac{1}{4\pi}\int\limits_L \left\{ \frac{[f_2^{+}(t) + f_2^{-}(t)]\, Y_0(z)}{Y_0^{+}(t)} + f_2^{+}(t) - f_2^{-}(t) \right\} T(t-z)\, dt + \tag{1.21}$$

$$+ F_3(z) + F_4(z), \quad Y_0(z) = Y^{-1}(z) \prod_{k=1}^{N} S(z-d_k)$$

$$F_3(z) = \sum_{k=1}^{Q} [A_{k1} + iA_{k2} Y_0(z)]\, T(z-t_k) + \frac{1}{2}\sum_{k=1}^{K} \{B_k [1 + Y_0^{-1}(c_{2k-1})\, Y(z)] \times$$

$$\times T(z - c_{2k-1}) + \bar{B}_k [1 - Y_0^{-1}(\bar{c}_{2k-1})\, Y(z)]\, T(z - \bar{c}_{2k-1})\}, \quad B_k = B_{k1} + iB_{k2}$$

$$F_4(z) = C_0 + \sum_{k=1}^{N}\sum_{j=1}^{p_k'} C_{kj} [T(z-b_k)]^{-1} + iQ_{N+S_0}(z)\, Y^{-1}(z) \prod_{k=1}^{N} [S(z-b_k)]^{-q_k'} \times$$

$$\times \prod_{n=1}^{R_0} S(z - a_n^*), \quad Q_N(z) = D_0 + \sum_{k=1}^{\frac{1}{2}N} \{D_{k1}\cos kz + D_{k2}\sin kz\}, \quad N = 2p$$

$$Q_N(z) = \sum_{k=0}^{\frac{1}{2}(N-1)} \{D_{k1}\cos(k + {}^1/_2)z + D_{k2}\sin(k + {}^1/_2)z\}, \quad N = 2p+1,$$

$$p = 0, 1, \ldots$$

$$p_k' = p_k, \ q_k' = q_k \ (b_k \,\bar{\in}\, M'); \ p_k' = q_k, \ q_k' = p_k - 1 \ (b_k \equiv b_k^* \in M^1)$$

$$p_k = E\{{}^1/_2(S_k + 1)\}, \ q_k = E\{{}^1/_2 S_k\}, \ S_0 = S - R_0 - R_1$$

Here a_n^* $(n = 1, \ldots, R_0)$ and b_n^* $(n = 1, \ldots, R_1)$ are the boundary points a_k and b_k of L that belong to M^1, d_k $(k = 1, \ldots, N)$ form the N-element set of notes of L, which contains all the points a_n^* and A_{kj}, B_{kj}, C_{kj}, C_0, D_0 are real constants.

The number of arbitrary constants in the resulting solution is equal to $2K + N + 2Q - R_0 - R_1 + S + 2$. $2K$ of these constants are required to remove the poles of $F(z)$ at $z = c_{2k-1}$ by solving the system of equations $F(C_{2k-1}) = 0$, where $k = 1, 2, \ldots, K$.

Therefore the general solution of the combined periodic Hilbert–Riemann problem contains $N + 2Q - R_0 - R_1 + S + 2$ arbitrary real constants. It corresponds to the highest degree of the integrable singularity at each node $z = d$ and has the form $\Phi(z) = O[(z-d)^{\zeta}]$ with $\operatorname{Re}\zeta \equiv -\frac{1}{2}$.

We remark that the general solution of the form (1.19), (1.21) and (1.14) contains the least possible number of arbitrary constants and so it involves the least number of equations in order to remove the poles.

2. The combined periodic Dirichlet–Riemann boundary-value problem is determined by (1.1) when $p(x) \equiv 1$, $x \in L$ and L^2 is not present. Then, as before, the canonical solution of the problem can be expressed by (1.14) with

$$h_0^{\pm}(t) = \pi n_k + \sum_{j=1}^{2K} (-1)^j \arg S(t - c_j), \quad t \in L \tag{2.1}$$

It has oscillating singularities for $z = s_k$ and a zero of order $\frac{1}{2} - i\gamma$ for $z = t_k$, and is bounded at $z = a_k$, $z = b_k$ and at infinity.

The general solution of the problem in the class of functions that have an integrable singularity at each node can be expressed by means of the formulas of Sec. 1 for $p_k = q_k = 0$. When the poles $z = c_{2k-1}$ are removed, there remain $N + 2Q - R_0 - R_1 + 2$ arbitrary constants.

3. There are two classes of problems in the theory of elasticity for orthotropic or isotropic elastic media under the conditions of statics or stationary subsonic motion which can be reduced to the periodic Hilbert–Riemann or Dirichlet–Riemann problem considered above. The former class includes problems of contact between an elastic half-plane and a periodic system of linked, partially or completely detached punches, and flexible cover-plates. The latter class includes problems concerned with deformations of an elastic plane that consists of two half-planes with distinct elastic characteristics weakened along the line of bonding by a periodic system of gaps. There are alternating segments of free surface and segments of full adhesion to the punches, slippage contact, and contact with smooth punches and inextensible filaments placed in arbitrary order on the edges of the gaps in a one-period strip.

For a finite number of punches, cover plates, and cuts forming the contours L and M, the above problems have been solved in [1–4, 8] by reduction to the Hilbert–Riemann problem for one or more complex potentials. The form of the solution of each of these problems can be preserved in the periodic case. It is then obvious that all the coefficients of the Hilbert–Riemann problems will remain the same and only the contours L and M will now be periodically repeated in (1.1) and (1.2).

As an illustration, we will consider the periodic contact problem for an elastic isotropic half-plane $-\infty < x < \infty$, $y < 0$ under static conditions. Let L_r, $r = 1, 2, 3$ be the systems of intervals $\langle a_{k2}, b_{k2} \rangle$, $k = 1, 2, \ldots, k_r$ on the real axis, let L_4 be the complement of $L_1 \cup L_2 \cup L_3$ to $[0, 2\pi)$, and let $L_k \cap L_l = 0$, for $k \neq l$. Let the half-plane be in full contact with the punches in L_1, in contact with slippage with the punches in L_2, and in contact with flexible cover-plates in L_3, and let the normal and shear stress be given on L_4.

We can write down the boundary conditions

$$u'(x) + iv'(x) = g_0(x), \; g_0(x) = g_{01}(x) + ig_{02}(x), \; x \in L_1 \tag{3.1}$$

$$v'(x) = v_0(x), \; \tau_{xy}(x) = 0, \; x \in L_2 \tag{3.2}$$

$$u'(x) = u_0(x), \; \sigma_y(x) = 0, \; x \in L_3 \tag{3.3}$$

$$\sigma_y(x) = \sigma_0(x), \; \tau_{xy}(x) = \tau_0(x), \; x \in L_4 \tag{3.4}$$

for the problem within one period. Here we assume that each of the given functions satisfies the Hölder condition. The apostrophe denotes a derivative with respect to x.

We shall seek a solution of the problem in Muskhelishvili's form [9]:

$$\sigma_y - i\tau_{xy} = \Phi(z) - \Phi(\bar{z}) + (z - \bar{z})\overline{\Phi'(z)} \tag{3.5}$$

$$2\mu(u' + iv') = \varkappa\Phi(z) + \Phi(\bar{z}) - (z - \bar{z})\overline{\Phi'(z)}$$

$$\Phi(z) = \tfrac{1}{4}(\sigma_x^\infty + \sigma_y^\infty) + 2i\mu(\varkappa + 1)^{-1}\varepsilon^\infty + o(1), \; z \to -i\infty \tag{3.6}$$

where $\varkappa = 3 - 4\nu$, ν and μ are Poisson's ratio and the shear modulus of the medium of the half-plane, σ_y^∞ and τ_{xy}^∞ are the normal and shear stresses, which penetrate to infinity, $\sigma_y^\infty - i\tau_{xy}^\infty = \frac{1}{2}\pi^{-1}(Y - iX)$, (X, Y) is the principal vector of the external forces applied to the boundary within the period $[0, 2\pi]$ and σ_x^∞ and ε^∞ are the tension and twist at infinity (prescribing the values of these coefficients is equivalent to posing the periodicity or quasiperiodicity condition for the displacements).

Substituting (3.5) into (3.1)–(3.4), we obtain the combined periodic Hilbert–Riemann problem (1.1), (1.2) for $\Phi(z)$, in which

$$M^1 = L_1,\ M^2 = L_4,\ L^1 = L_2,\ L^2 = L_3,\ G = \varkappa$$
$$g(x) = 2\mu g_0(x),\ x \in M^1;\ g(x) = -\sigma_0(x) + i\tau_0(x),\ x \in M^2$$
$$f^{\pm}(x) = 2\mu(\varkappa + 1)^{-1} v_0(x),\ x \in L^1;\ f^{\pm}(x) = 2\mu(\varkappa + 1)^{-1} u_0(x),\ x \in L^2$$

In the absence of condition (3.2) or (3.3), the boundary-value problem (1.1), (1.2) reduces to the combined Dirichlet–Riemann problem.

The values of the principal vectors of the normal and shear forces applied to the punches and cover-plates, as well as the magnitudes of the jumps of the displacements (the tension) between them, serve as additional conditions which determine the solution of the problem of the theory of elasticity (3.1)–(3.4) according to the nature of the contact between the punches and cover-plates. The total number of these parameters, including the characteristics of the stress and strain state of the half-plane at infinity is obviously equal to the number of arbitrary constants in the solution of the corresponding combined boundary-value problem.

4. *Example 1.* Suppose that, within one period, there are two flat punches that are in contact with the boundary of the half-plane. Let one of the punches be attached to the plane in $[a, b]$, while the other one is in a state of contact with slippage in $[c, d]$. The boundary conditions for this problem can be described by (3.1), (3.2) and (3.4) for $g_0(x) = v_0(x) = \sigma_0(x) = \tau_0(x) \equiv 0$, $L_1 = [a, b]$, and $L_2 = [c, d]$. The periodic version of the problem has been discussed in [2].

The canonical solution of the resulting Dirichlet–Riemann problem can be constructed from (1.14) and (2.1) for $N = 1$, $s_1 = a$, $t_1 = b$, $a_1 = c$, $b_1 = d$, $n_1 = 0$. We have

$$X(z) = e^{i\varphi(z)} \sqrt{\frac{S(z-b)}{S(z-a)}},\quad \varphi(z) = -\frac{\gamma Y(z)}{2} \int_a^b \frac{dt}{Y(t)\, S(t-z)}$$
$$Y(z) = \sqrt{S(z-c) S(z-d)},\quad Y(t) = -\sqrt{S(c-t) S(d-t)},\quad t \in [a, b] \tag{4.1}$$

Evaluating the integral in (2.7) by reducing it to an integral in the $w = e^{iz}$ plane that is listed in tables, after some reduction we get

$$\varphi(z) = 2\gamma \ln \left[\frac{\sqrt{S(d-b) S(z-c)} + \sqrt{S(c-b) S(z-d)}}{\sqrt{S(d-a) S(z-c)} + \sqrt{S(c-a) S(z-d)}} \sqrt{\frac{S(z-a)}{S(z-b)}} \right] \tag{4.2}$$

Next, according to Sec. 2 and (1.19), (1.21), we have

$$\frac{\Phi(z)}{X(z)} = A_{11} + iA_{12} \sqrt{\frac{S(z-d)}{S(z-c)}} + C_0 + \frac{i(D_{11} \cos {}^1\!/_2 z + D_{12} \sin {}^1\!/_2 z)}{\sqrt{S(z-c) S(z-d)}} \tag{4.3}$$

We substitute (4.3) into (4.1). On applying some trigonometric transformations, we can represent the general solution in the form

$$\Phi(z) = \frac{e^{i\varphi(z)}}{\sqrt{S(z-a) S(z-b)}} \left[P(z) + \frac{iQ(z)}{\sqrt{S(z-c) S(z-d)}} \right] \tag{4.4}$$

$$P(z) = A_1 \cos {}^1\!/_2 (z - a_*) + A_2 \sin {}^1\!/_2 (z - a_*),\quad a_* = {}^1\!/_2 (a + b) \tag{4.5}$$
$$Q(z) = C_1 \cos(z - c_*) + C_2 \sin(z - c_*) + C_3,\quad c_* = {}^1\!/_4 (a + b + c + d)$$

where A_j and C_j are new arbitrary real constants.

In accordance with (4.4), (4.5), (4.2) and (3.5), we write down the values for the normal stress in the slippage section:

$$\sigma_y(x) = -\frac{2}{\sqrt{S(x-a) S(x-b)}} \left[P(x) \operatorname{sh} \varphi_0(x) + \frac{Q(x) \operatorname{ch} \varphi_0(x)}{\sqrt{S(x-c) S(d-x)}} \right] \tag{4.6}$$
$$\varphi_0(x) = \gamma \left[\operatorname{arctg} \sqrt{\frac{S(c-a) S(d-x)}{S(d-a) S(x-c)}} - \operatorname{arctg} \sqrt{\frac{S(c-b) S(d-x)}{S(d-b) S(x-c)}} \right]$$

where, obviously, $\varphi_0(x) \geq 0$ for $x \in [c, d]$.

We now determine the arbitrary constants. By (3.5) and (3.6), the conditions at infinity give rise to the following system of linear algebraic equations with respect to A_1, A_2, C_1, C_2:

$$[iA_1 + A_2 - 2(iC_1 + C_2)] \exp(i\beta_0) = {}^1/_4(\sigma_x^\infty + \sigma_y^\infty) + 2i\mu(\varkappa + 1)^{-1}\varepsilon^\infty \tag{4.7}$$

$$[iA_1 + A_2 + 2(iC_1 + C_2)] \exp(-i\beta_0) = {}^1/_4\sigma_x^\infty - {}^3/_4\sigma_y^\infty - 2i\mu(\varkappa + 1)^{-1}\varepsilon^\infty - i\tau_{xy}^\infty$$

$$\beta_0 = \varphi(-i\infty) = 2\gamma \ln[(\sqrt{e^{id} - e^{ib}} + \sqrt{e^{ic} - e^{ib}})(\sqrt{e^{id} - e^{ia}} + \sqrt{e^{ic} - e^{ia}})^{-1}] \tag{4.8}$$

If the punches are connected with one another, then C_3 can be found from the condition

$$\int_b^c v'(x)\,dx = 0 \tag{4.9}$$

But if the punches can move independently without rotation and the external forces (X, Y_1) and Y_2 applied to $[a, b]$ and $[c, d]^2$, respectively, are known, then one must integrate expression (4.6) for the contact stress under a slippage punch in order to determine C_3:

$$\int_c^d \sigma_y(x)\,dx = Y_2 \tag{4.10}$$

Finally, we remark that the mechanical feasibility condition $\sigma_y \leqslant 0$ for $x \in [c, d]$ imposes restrictions on the parameters of the geometry and the forces involved in the problem.

5. *Example 2.* Let the end-points of L_1 and L_2 coincide with one another, which corresponds to the case when a periodic system of individual punches with detached edges is in contact with a half-plane.

The canonical solution of the combined boundary-value problem can be expressed by the same formula (4.1), in which we obtain

$$\varphi(z) = 2\gamma \ln[\sqrt{S(d-b)S(z-a)}(\sqrt{S(d-a)S(z-c)} + \\ + \sqrt{S(b-a)S(z-d)})^{-1}] \tag{5.1}$$

$$\beta_0 = \varphi(-i\infty) = \gamma \ln[\sqrt{e^{id} - e^{ib}}(\sqrt{e^{id} - e^{ia}} + \sqrt{e^{ib} - e^{ia}})^{-1}] \tag{5.2}$$

instead of (4.2) and (4.8) for $c = b$.

Similar to (4.4), we represent the general solution of the problem in the form

$$\Phi(z) = \frac{e^{i\varphi(z)}}{\sqrt{S(z-a)}}\left[\frac{P(z)}{\sqrt{S(z-b)}} + \frac{iQ(z)}{\sqrt{S(z-d)}}\right] \tag{5.3}$$

$$P(z) = A_1 \cos{}^1/_2(z - a_*) + A_2 \sin{}^1/_2(z - a_*),\quad a_* = {}^1/_2(a + b)$$
$$Q(z) = C_1 \cos{}^1/_2(z - b_*) + C_2 \sin{}^1/_2(z - b_*),\quad b_* = {}^1/_2(c + d) \tag{5.4}$$

The arbitrary constants can be determined from the conditions at infinity. From (5.3), (5.4), and (3.6) we get

$$(V_1 - V_2)\exp(i\beta_0) = {}^1/_4(\sigma_x^\infty + \sigma_y^\infty) + 2i\mu(\varkappa + 1)^{-1}\varepsilon^\infty,\quad V_1 = iA_1 + A_2 \tag{5.5}$$

$$(V_1 + V_2)\exp(-i\beta_0) = {}^3/_4\sigma_y^\infty - {}^1/_4\sigma_x^\infty + 2i\mu(\varkappa + 1)^{-1}\varepsilon^\infty + i\tau_{xy}^\infty,\quad V_2 = C_1 - iC_2$$

Let there be only a vertical force applied to each punch and let there be no field at infinity. Then $\sigma_y^\infty = \frac{1}{2}\pi^{-1}Y$, $\tau_{xy}^\infty = \varepsilon^\infty = \sigma_x^\infty = 0$, and, solving (5.5), we can find that ($k = 1, 2$):

$$8\pi V_k = -w_k Y,\quad w_1 = \cos\beta_0 + 2i\sin\beta_0,\quad w_2 = 2\cos\beta_0 + i\sin\beta_0 \tag{5.6}$$

We write down the expressions for the contact stress in the slippage section $x \in [b, d]$:

$$\sigma_y(x) = \frac{Y}{4\pi\sqrt{S(x-a)}}\left[\frac{\operatorname{Im} W_1(x)\operatorname{sh}\varphi_0(x)}{\sqrt{S(x-b)}} + \frac{\operatorname{Re} W_2(x)\operatorname{ch}\varphi_0(x)}{\sqrt{S(d-x)}}\right]$$

$$\varphi_0(x) = \gamma \operatorname{arctg}\{[S(b-a)S(d-x)]^{1/2}[S(d-a)S(x-b)]^{-1/2}\},\quad 0 \leqslant \varphi_0(x) \leqslant {}^1/_2\pi\gamma$$

$$W_1(x) = w_1 \exp[{}^1/_2 i(x - a_*)],\quad W_2(x) = w_2 \exp[{}^1/_2 i(b_* - x)] \tag{5.7}$$

and in the adhesion section $x \in [a, b]$:

$$\sigma_y - i\tau_{xy} = \frac{(\varkappa + 1)Ye^{i\varphi_0(x)}}{8\pi\sqrt{\varkappa S(x-a)}}\left[\frac{i\operatorname{Im} W_1(x)}{\sqrt{S(b-x)}} - \frac{\operatorname{Re} W_2(x)}{\sqrt{S(d-x)}}\right] \tag{5.8}$$

The asymptotic form for either of these expressions at any nodal point has the form

$$\sigma_y = \sigma_0 + O(\sqrt{b-x}),\ \tau_{xy} = K_{IIb}[2\pi(b-x)]^{-1/2} + O(\sqrt{b-x}),\ x \to b-0 \tag{5.9}$$
$$\sigma_y = K_{Ib}[2\pi(x-b)]^{-1/2} + \sigma_0 + O(\sqrt{x-b}),\ x \to b+0$$
$$\sigma_y = K_{Id}[2\pi(d-x)]^{-1/2} + O(\sqrt{d-x}),\ x \to d-0$$

$$K_{Ib} = \frac{K_{IIb}(\varkappa-1)}{\varkappa+1},\quad \sigma_0 = \frac{(\varkappa+1)Y\rho_2\cos[\theta_2+{}^1/_2(d-a_*)]}{8\pi\sqrt{\varkappa S(b-a)S(d-b)}} \tag{5.10}$$
$$K_{IIb} = \frac{(\varkappa+1)Y\rho_1\sin[\theta_1+{}^1/_4(b-a)]}{4\sqrt{\pi\varkappa S(b-a)}},\quad K_{Id} = \frac{Y\rho_2\cos[\theta_2-{}^1/_4(d-a)]}{2\sqrt{\pi S(d-a)}}$$
$$\rho_k e^{i\theta_k} = w_k,\ k=1,2$$

It follows from (5.8) that the contact stress is compressing in the entire slippage section $[c,d]$ if the inequalities

$$Y\,\mathrm{Im}\,W_1(x) \leqslant 0,\ Y\,\mathrm{Re}\,W_2(x) \leqslant 0 \tag{5.11}$$

are satisfied. According to (5.10), the conditions $K_{Ib} \leqslant 0$ and$K_{Id} \leqslant 0$, which are necessary for (5.11), lead to the relations

$$Y\sin[\theta_1 + {}^1/_4(b-a)] \leqslant 0,\ Y\cos[\theta_2 + {}^1/_4(d-a)] \leqslant 0 \tag{5.12}$$

Conditions (5.12) will obviously be sufficient if the punches are short so that $d-a \ll 2\pi$. Here, by virtue of Saint-Venant's principle, one can apply the results of [2], where a rigorous proof of the sufficiency is given.

Nevertheless, the qualitative features of the solution mentioned in the discussion of the non-periodic case in [2] are also preserved in the general case of a periodic problem.

If the length of the adhesion section is fixed, the trigonometric inequalities (5.11) restrict the possible length of the single contact section beneath the detached edge of a punch to the values from a denumerable set of descending intervals.

If the inequalities (5.11) or (5.12) are not satisfied, a contact occurs in two or more sections of the part of the punch with a notch.

The total length of the part with a notch can exceed that of the slippage section. Moreover, if $K_{Id} = 0$, smooth adhesion of the contact surfaces occurs at $x = d$.

According to (5.10) and (5.12), the normal stress intensity factor K_{Ib} is non-positive. Thus the gap of detachment can propagate only due to the shear stress. As it develops, the gap passes through a denumerable set of states characterized by the relations $K_{IIb} = K_{Ib} = 0$. If the lengths of the slippage and adhesion sections satisfy this condition, the gap is stable and its subsequent development can occur only due to some non-elastic factors (for example, plasticity, temperature, or corrosion).

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